* UNIT - I *

Arithmetic functions and Dirichlet functions

Arithmetic :-

A real (or) complex valued functions defined on the set of positive integers is called an arithmetic function.

Mobius function(μ):-

The mobius function $\mu(n)$ is defined as follows.

(i) If $n=1 \Rightarrow \mu(1)=1$

(ii) If $n>1$, we can write $n=p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where $p_1, p_2 \cdots p_k$ are prime numbers then $\mu(n)=(-1)^k$ if $a_1, a_2 \cdots a_k = 1$

$\mu(n)=0$, otherwise

n	1	2	3	4	5	6	7	8	9	10
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1

If $n=1$, $\mu(1)=1$

$n=2$, $\mu(2)=2^1=(-1)^1=-1$

$n=3$, $\mu(3)=3^1=(-1)^3=-1$

$n=4$, $\mu(4)=2^2=0$

$n=5$, $\mu(5)=5^1=(-1)^1=-1$

$n=6$, $\mu(6)=\mu(2\times 3)=2^1\times 3^1=(-1)^2=1$

$n=7$, $\mu(7)=7^1=(-1)^1=-1$

$n=8$, $\mu(8)=2^3=0$

$n=9$, $\mu(9)=3^2=0$

$n=10$, $\mu(10)=2^1\cdot 5^1=(-1)^2=1$

* Theorem:- If $n>1$, we have $\sum_{d/n} \mu(d) = \left[\frac{1}{n}\right]$

$$= \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

Proof:- If $n=1$

we have $\mu(1)=1$

For $n=1$, $\sum_{d/1} \mu(d) = 1$

$\therefore$ For $n=1$, the result is true

Suppose $n>1$,

write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

[In the summation $\sum_{d/n} \mu(d)$ the only non-zero terms from $d=1$ and from those divisors of n which are product of distinct primes]

Consider

$1, p_1, p_2, \cdots p_k, p_1p_2, p_2p_3, \cdots p_{k-1}p_k, p_1p_2p_3, \cdots p_1p_2p_3 \cdots p_k$

$\sum_{d/n} \mu(d) = \mu(1) + \mu(p_1) + \mu(p_2) + \cdots + \mu(p_k) + \mu(p_1p_2) + \mu(p_2p_3) + \cdots + \mu(p_{k-1}p_k) + \mu(p_1p_2p_3) + \cdots + \mu(p_1p_2p_3 \cdots p_k)$

$= 1 + (-1)^k + (-1)^k + \cdots + (-1)^k$

$= (1-1)^k$

$= 0$

$\therefore$ For $n>1$, we have $\sum_{d/n} \mu(d) = \left[\frac{1}{n}\right]$

$$= \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

Euler totient function:

If $n>1$, the euler totient function $\phi(n)$ is defined to the positive integers non-exceeding 'n' which are relatively prime to 'n'.

i.e., $\phi(n) = \sum_{\substack{k=1 \\ (n,k)=1}}^{n} 1$ (or)

$\phi(n) = \sum_{k=1}^{n} \left[\frac{1}{(n,k)}\right]$

$(n,k)=1$ ⟶ relatively prime

Note:-

(i) $\phi(1)=1$

(ii) $\phi(n)=n-1$ if 'n' is a factor

(iii) $\phi(n^\alpha) = n^\alpha\left[1-\frac{1}{n}\right]$, if 'n' is a prime

n	1	2	3	4	5	6	7	8	9	10
$\phi(n)$	1	1	2	2	4	2	6	4	6	4

If $n=1$, $\phi(1)=1$

$n=2$, $\phi(2)=2-1=1$

$n=3$, $\phi(3)=3-1=2$

$n=4$, $\phi(4)=\phi(2^2)=2^2\left[1-\frac{1}{2}\right]=4\left[\frac{1}{2}\right]=2$

$n=5$, $\phi(5)=5-1=4$

$n=6$, $\phi(6)=\phi(2\times3)=1\times2=2$

$n=7$, $\phi(7)=7-1=6$

$n=8$, $\phi(8)=\phi(2^3)=2^3\left[1-\frac{1}{2}\right]=2^3\left(\frac{1}{2}\right)=4$

$n=9$, $\phi(9)=\phi(3^2)=3^2\left[1-\frac{1}{3}\right]=9\left[\frac{2}{3}\right]=6$

$n=10$, $\phi(10)=\phi(2\times5)=1\times4=4$

Problems:-

1. Find the value of $\phi(126)$

$\phi(126) = 2^1 \times 3^2 \times 7^1$

$= \phi(2)\,\phi(3^2)\,\phi(7)$

$= 1 \cdot 6 \cdot 6$

$= 36$

2. $\phi(3600)$

$\phi(3600) = 2^4 \times 5^2 \times 3^2$

$= 8 \times 6 \times 20$

$= 48 \times 20$

$= 960$

3. $\phi(490)$

$\phi(490) = \phi(5)\,\phi(2)\,\phi(7^2)$

$= 168$

4. $\phi(728)$

$\phi(728) = 2^3 \times 7^1 \times 13^1$

$= 4 \cdot 6 \cdot 12$

$= 24(12)$

$= 288$

5. $\phi(768)$

$\phi(768) = 2^8 \times 3^1$

$= \phi(2^8)\,\phi(3)$

$= 128 \times 2$

$= 256$

6. $\phi(25200)$

$\phi(25200) = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$

$= 2^3 \cdot 6 \cdot 20 \cdot 6$

$= 8 \cdot 20 \cdot 36$

$= 160 \cdot 36$

$= 5760$

Theorem: A relation connecting ϕ & μ:

If $n \geq 1$, we have $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.

Proof: From the definition of quotient function, we have $\phi(n) = \sum_{k=1}^{n} 1$ (or) $\phi(n) = \sum_{k=1}^{n} \frac{1}{(n,k)}$ — (1)

Recall:

If $n>1$, we have $\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$ — (2)

By the above recall, we have

$$\phi(n) = \sum_{k=1}^{n} \sum_{d|(n,k)} \mu(d)$$

$$= \sum_{k=1}^{n} \sum_{\substack{d|n \\ d|k}} \mu(d) \quad \text{— (3)}$$

For a fixed division d (or) k we must sum over all k values are in the range $1 \leq k \leq n$ then $k = 1, 2 \cdots n$

since $k = qd$

we know that $k = 1, 2, 3 \cdots n$

$$qd = 1, 2 \cdots n$$

$$q = 1/d, 2/d \cdots n/d$$

In eq (3) $\Rightarrow \phi(n) = \sum_{d|n} \sum_{q=1|d}^{n|d} \mu(d)$

$$= \sum_{d|n} \mu(d) \sum_{q=1|d}^{n|d} 1$$

$$= \sum_{d|n} \mu(d) \frac{n}{d}$$

Hence $\phi(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d}$

Second product formula for $\phi(n)$

Theorem:

For $n \geq 1$, p is any prime then $\phi(n) = n \prod_{p/n} \left[1 - \frac{1}{p}\right]$

Proof:

For $n = 1$, there are no prime numbers which divides '1'.

$\therefore n \prod_{p/n} \left[1 - \frac{1}{p}\right] = 1$

we know that $\phi(1) = 1$

$\therefore \phi(n) = n \prod_{p/n} \left[1 - \frac{1}{p}\right]$

$\therefore$ The result is true for $n = 1$

Suppose $n > 1$, we have $\phi(n) = \sum_{d/n} \mu(d) \cdot \frac{n}{d}$

$$= n \sum_{d/n} \frac{\mu(d)}{d} \longrightarrow (1)$$

Let $P_1, P_2, \ldots, P_k$ be the distinct prime divisors of 'n'

Then the product can be written as

$$\prod_{p/n} \left[1 - \frac{1}{p}\right] = \prod_{i=1}^{k} \left[1 - \frac{1}{P_i}\right]$$

$$= \left[1 - \frac{1}{P_1}\right]\left[1 - \frac{1}{P_2}\right] \cdots \left[1 - \frac{1}{P_k}\right]$$

$$= \left[1 - \frac{1}{P_1} - \frac{1}{P_2} + \frac{1}{P_2 P_1}\right]\left(1 - \frac{1}{P_3}\right) + \cdots \left(1 - \frac{1}{P_k}\right)$$

$$= \left[1 - \frac{1}{P_1} - \frac{1}{P_2} - \frac{1}{P_3} + \frac{1}{P_1 P_2} + \frac{1}{P_2 P_3} + \frac{1}{P_1 P_3} - \frac{1}{P_1 P_2 P_3} \cdots \left[1 - \frac{1}{P_k}\right]\right.$$

$$= 1 + \sum_{i=1}^{k} \frac{(-1)}{P_i} + \sum_{i,j=1}^{k} \frac{(-1)^2}{P_i P_j} + \cdots + \frac{(-1)^k}{P_1 P_2 \cdots P_k} \longrightarrow$$

Each term on the R.H.S of eq^n (2) is of the form $\frac{\pm 1}{d}$, where 'd' is the divisor of 'n'

$\therefore$ The numerator $\pm 1 = \mu(d)$

∴ From ②, we have $\prod_{p/n}\left[1-\frac{1}{p}\right] = \sum_{d/n}\frac{\mu(d)}{d}$ → ③

Substitute ③ in ①, we get $\phi(n) = n\prod_{p/n}\left[1-\frac{1}{p}\right]$

Hence the proof.

Theorem:-

Euler totient function on the following properties

(i) $\phi(p^\alpha) = p^\alpha - p^{\alpha-1}$ ∀ primes p ∀ $\alpha \geq 1$

(ii) $\phi(m,n) = \phi(m)\,\phi(n)\frac{d}{\phi(d)}$ where $d = (m,n)$

(iii) $\phi(m,n) = \phi(m)\,\phi(n)$ if $(m,n)=1$

(iv) $a/b \Rightarrow \phi(a)/\phi(b)$

Proofs:-

(i) From the product formula, we have

$$\phi(n) = n\prod_{p/n}\left[1-\frac{1}{p}\right] \longrightarrow ①$$

let $n = p^\alpha$ where 'p' is a prime and $\alpha \geq 1$

then $\phi(p^\alpha) = p^\alpha\left[1-\frac{1}{p}\right]$

$= p^\alpha - p^{\alpha-1}$

(ii) From ① we have $\frac{\phi(n)}{n} = \prod_{p/n}\left[1-\frac{1}{p}\right] \longrightarrow ②$

let $d = \gcd(m,n)$

consider $\frac{\phi(m,n)}{mn} = \prod_{p/mn}\left[1-\frac{1}{p}\right]$

Each prime divisor of mn is either a prime divisor of m (or) n and also it divides gcd(m,n)

Hence $\frac{\phi(mn)}{mn} = \frac{\prod_{p/m}\left[1-\frac{1}{p}\right]\prod_{p/n}\left[1-\frac{1}{p}\right]}{\prod_{p/d}\left[1-\frac{1}{p}\right]}$

$\Rightarrow \frac{\phi(mn)}{mn} = \frac{\frac{\phi(m)}{m}\cdot\frac{\phi(n)}{n}}{\frac{\phi(d)}{d}}$

$$\Rightarrow \frac{\phi(mn)}{mn} = \frac{d\,\phi(m)\phi(n)}{mn\,\phi(d)}$$

$$\therefore \phi(mn) = \frac{d \cdot \phi(m)\phi(n)}{\phi(d)}$$

$$\phi(m,n) = \phi(m)\phi(n)\frac{d}{\phi(d)}$$

(iii) Suppose $(m,n)=1$

put $d=(m,n)=1$.

From the property ②, we have

$$\phi(m,n) = \phi(m)\cdot\phi(n)\frac{d}{\phi(d)}$$

$$= \phi(m)\phi(n)\frac{1}{\phi(d)}$$

$$\therefore \phi(m,n) = \phi(m)\phi(n)$$

(iv) $a/b \Rightarrow (a,b) = a$

From condition ② we have $d=(m,n)$

$$\Rightarrow \phi(m,n) = \phi(m)\phi(n)\cdot\frac{d}{\phi(d)}$$

$$\Rightarrow \phi(a,b) = \phi(a)\cdot\phi(b)\frac{a}{\phi(a)}$$

$$\phi(b) = \phi(a)\left[\frac{\phi(a,b)}{a\cdot\phi(a)}\right]$$

$$\Rightarrow \phi(a)/\phi(b)$$

$$\therefore \phi(a)/\phi(b)$$

Sec: The Dirchilet Product of arithmetic Functions

let f, g be two arithmetic functions then we define their dirchilet product as

$$(f*g)(n) = \sum_{d/n} f(d)\, g\left[\frac{n}{d}\right]$$

Result:- Prove that $\phi = \mu * N$

Proof:- $\phi(n) = \sum_{d/n} \mu(d) \frac{n}{d}$

$N\left(\frac{n}{d}\right) = \frac{n}{d}$ where 'N' is identity-function

$$\phi(n) = \sum_{d/n} \mu(d) \cdot N\left(\frac{n}{d}\right)$$

$$\phi(n) = (\mu * N)(n)$$

$$\therefore \phi = \mu * N$$

Theorem:-

Dirchilet multiplication is commutative and Associative.

Proof:- let 'f' and 'g' be two arithmetic functions

Commutative:-

To s.T $f * g = g * f$

consider $(f * g)(n) = \sum_{a/n} f(a) g\left[\frac{n}{a}\right]$

$$= \sum_{n=ab} f(a) g\left[\frac{n}{a}\right]$$

$$= \sum_{b/n} f\left[\frac{n}{b}\right] g(b)$$

$$= \sum_{b/n} g(b) f\left[\frac{n}{b}\right]$$

$$= (g * f)(n)$$

$$\therefore f * g = g * f$$

Associative:-

let f, g, h be any three arithmetic functions

then we have to s.T $f * (g * h) = (f * g) * h$

let $A = g * h$

consider $f * (g * h)(n) = (f * A)(n)$

$$= \sum_{a/n} f(a) A\left[\frac{n}{a}\right]$$

$$= \sum_{n=ad} f(a)(g*h)(d)$$

$$= \sum_{n=ad} f(a) \sum_{b|d} g(b) h\left(\frac{d}{b}\right)$$

$$= \sum_{n=ad} f(a) \sum_{d=bc} g(b) h(c)$$

$$= \sum_{n=abc} f(a) g(b) h(c) \longrightarrow (1)$$

Now let $f * g = A$

consider $((f*g)*h)(n) = (A*h)(n)$

$$= \sum_{d|n} A(d) h\left[\frac{n}{d}\right]$$

$$= \sum_{n=dc} (f*g)(d) h(c)$$

$$= \sum_{n=dc} \sum_{a|d} f(a) g\left[\frac{d}{a}\right] h(c)$$

$$= \sum_{n=dc} \sum_{d=ab} f(a) g(b) h(c)$$

$$= \sum_{n=abc} f(a) g(b) h(c) \longrightarrow (2)$$

From (1) and (2), we have

$$f*(g*h) = (f*g)*h$$

Identity function:-

An arithmetic function $I(n)$ is said to be identity function is defined as

$$I(n) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if otherwise} \end{cases}$$

Note:- $\sum_{d|n} \mu(d) = I(n) = \left[\frac{1}{n}\right]$

theorem-7:

For all 'f' we have $I * f = f * I = f$

Proof:- For all f we have $I * f = f * I = f$

Consider $(f * I)(n) = \sum_{d|n} f(d) I\left[\frac{n}{d}\right]$

$$= \sum_{d|n} f(d) \left[\frac{1}{n/d}\right]$$

$$= \sum_{d|n} f(d) \left[\frac{d}{n}\right]$$

$$= \sum_{d=n,1} f(d) \left[\frac{d}{n}\right] + \sum_{\substack{d|n \\ d<n}} f(d) \left[\frac{d}{n}\right]$$

$$= f(n)(1) + \sum_{d<n} f(d) \cdot 0$$

$$= f(n)$$

$\therefore f * I = f$

Similarly, we can prove that $I * f = f$

$\therefore I * f = f * I = f$

Dirchilative Inverse:- If 'f' is an arthematic function with $f(1) \neq 0$ there is a unique arithmetic function f^{-1} is called as Dirchiletive inverse of f $\Rightarrow f * f^{-1} = f^{-1} * f = I$

(i) $f^{-1}(1) = \frac{1}{f(1)}$ if $n = 1$

(ii) $f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left[\frac{n}{d}\right] f^{-1}(d)$

Note:-

1. $(f * g)(1) = f(1) \cdot g(1)$
2. $\sum_{d|n} \mu(d) = I(n) = 1|n$

3. $\sum_{d|n} f\left[\frac{n}{d}\right] f^{-1}(d) = \sum_{d=n} f\left[\frac{n}{d}\right] f^{-1}(d) + \sum_{\substack{d|n \\ d<n}} f\left[\frac{n}{d}\right] f^{-1}(d)$

Dirchiletive inversion theorem:

Statement: if f is an arithmetic function with $f(1) \neq 0$ there is a unique arithmetic function f^{-1} such that $f * f^{-1} = f^{-1} * f = I$. Moreover f^{-1} is given by the formula

(i) $f^{-1}(1) = \frac{1}{f(1)}$ if $n=1$

(ii) $f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left[\frac{n}{d}\right] f^{-1}(d)$

Proof:

(i) We know that $f * f^{-1} = I$

$$(f * f^{-1})(n) = I(n)$$

$$(f * f^{-1})(1) = I(1) \quad [\because n=1]$$

$$f(1) \cdot f^{-1}(1) = 1$$

$$f^{-1}(1) = \frac{1}{f(1)}$$

(ii) we know that $(f^{-1} * f)(n) = I(n)$

$$\Rightarrow \sum_{d|n} f\left[\frac{n}{d}\right] f^{-1}(d) = 0 \quad [\because n>1]$$

$$\Rightarrow \sum_{d=n} f\left[\frac{n}{d}\right] f^{-1}(d) + \sum_{\substack{d|n \\ d<n}} f\left[\frac{n}{d}\right] f^{-1}(d) = 0$$

$$\Rightarrow f(1) f^{-1}(n) + \sum_{\substack{d|n \\ d<n}} f\left[\frac{n}{d}\right] f^{-1}(d) = 0$$

$$\Rightarrow f(1) f^{-1}(n) = -\sum_{\substack{d|n \\ d<n}} f\left[\frac{n}{d}\right] f^{-1}(d)$$

$$\therefore f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left[\frac{n}{d}\right] f^{-1}(d)$$

Result:- μ and u are dirchiletive inverse of each other $u = \mu^{-1}$ and $\mu = u^{-1}$

Proof:-

we know that $I(n) = \sum_{d|n} \mu(d)$

$$= \sum_{d|n} \mu(d) \cdot 1$$

$$= \sum_{d|n} \mu(d)\, u\left[\frac{n}{d}\right] \quad \left[\because u\left[\frac{n}{d}\right] = 1\right]$$

$$I(n) = (\mu * u)(n)$$

$$\therefore I = \mu * u$$

$$\because u = \mu^{-1} \text{ and } \mu = u^{-1}$$

Hence the proof

Unit function:- Unit function(u) to be the arithemetic function such that $u(n) = 1$

Note:-

$$\sum_{d|n} \mu(d) = I(n) = \frac{1}{n}$$

Mobius Inverse formula

The equation $f(n) = \sum_{d|n} g(d) \Leftrightarrow g(d) = \sum_{d|n} f(d)\mu\left[\frac{n}{d}\right]$

Proof:-

Part 1:-

Suppose that $f(n) = \sum_{d|n} g(d)$

$$= \sum_{d|n} g(d) \cdot 1$$

$$= \sum_{d|n} g(d) \cdot \mu\left[\frac{n}{d}\right]$$

$$f(n) = (g * \mu)(n)$$

$$f = g * u$$

$$\Rightarrow f * \mu = g * u * \mu$$

$$= g * (\mu * \mu) \quad [\because \mu * \mu = I]$$

$= g * I$

$= g$

$\Rightarrow f * \mu = g$

$\because g(n) = (f * \mu)(n)$

$\therefore g(n) = \sum_{d|n} f(d)\mu\left[\frac{n}{d}\right]$

Part 2:- Suppose that $g(n) = \sum_{d|n} f(d)\cdot\mu\left(\frac{n}{d}\right)$

$\therefore g = f * \mu$

$\Rightarrow g * \mu = f * \mu * \mu$

$= f * (\mu * \mu)$

$= f * I = f$

$\Rightarrow g * \mu = f$

$\Rightarrow g * \mu(n) = f(n)$

$\Rightarrow f(n) = \sum_{d|n} g(d)\mu\left[\frac{n}{d}\right]$

$= \sum_{d|n} g(d)\cdot 1 \qquad \left[\because \mu\left[\frac{n}{d}\right] = 1\right]$

$\therefore f(n) = \sum_{d|n} g(d)$

Man-goldt function $\Lambda(n)$:- for every integer $n \geq 1$ we have

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^m, \text{ for some 'p' is prime and } m \geq 1 \\ 0, \text{ otherwise} \end{cases}$$

n	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$	0	$\log 2$	$\log 3$	$\log 2$	$\log 5$	0	$\log 7$	$\log 2$	$\log 3$	0

In there is a basis of all primes and powers are ≥ 1. so, defined its value equal to $\log 2, \log 3, \log 2, \log 5, \log 7, \log 2, \log 3$ and the remaining cases the value is equal to zero.

Theorem:-

If $n \geq 1$ we have $\Lambda(n) = \sum_{d|n} \Lambda(d)$ if $n \neq 1$ (log n)

proof:- if $n = 1$

$L.H.S \Rightarrow \log(n) = \log(1) = 0$

$R.H.S \Rightarrow \sum_{d|n} \Lambda(d) = \Lambda(1) = 0$

$\therefore L.H.S = R.H.S$

if $n > 1$

we know that $n = P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r}$

$$\log n = \log\left[P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r}\right]$$

$$\log n = \log P_1^{a_1} + \log P_2^{a_2} + \cdots + \log P_r^{a_r}$$

$$= a_1 \log P_1 + a_2 \log P_2 + \cdots + a_r \log P_r$$

$$\log n = \sum_{k=1}^{r} a_k \log P_k \longrightarrow (1)$$

Now Consider $\sum_{d|n} \Lambda(d)$, where $n = P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r}$

i.e., $\sum_{d|n} \Lambda(d) = \Lambda(P_1^1) + \Lambda(P_1^2) + \cdots + \Lambda(P_1^{a_1}) +$

$\Lambda(P_2^1) + \Lambda(P_2^2) + \cdots + \Lambda(P_2^{a_2}) + \cdots$

$\cdots + \Lambda(P_r^1) + \Lambda(P_r^2) + \cdots + \Lambda(P_r^{a_r})$

$$= \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(P_k^m)$$

$$= \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log P_k$$

$$= \sum_{k=1}^{r} (\log P_k) \sum_{m=1}^{a_k} (1)$$

$$\sum_{d|n} \Lambda(d) = \sum_{k=1}^{r} \log P_k \cdot a_k \longrightarrow (2)$$

From (1) and (2), we have

$$\log n = \sum_{d|n} \Lambda(d)$$

Theorem:- If $n \geq 1$ then (i) $\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}$

(ii) $\Lambda(n) = -\sum_{d|n} \mu(d) \log d$

Proof:-

(i) we know that $\log n = \sum_{d|n} \Lambda(d)$

By above theorem, $f(n) = \sum_{d|n} g(n) \Leftrightarrow$

$$\sum_{d|n} f(d)\mu\left(\frac{n}{d}\right) = g(n)$$

let $f(n) = \log n$

$\Lambda(n) = g(n)$

then $\Lambda(n) = \sum_{d|n} \log d\, \mu\left(\frac{n}{d}\right)$

$$= \sum_{d|n} \mu(d) \log\left[\frac{n}{d}\right]$$

(ii) From (i),

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} [\log n - \log d]\mu(d)$$

$$= \sum_{d|n} \log n \mu(d) - \sum_{d|n} \log d \cdot \mu(d)$$

$$= \log n \left[\frac{1}{d}\right] - \sum_{d|n} \log d\, \mu(d)$$

$$= \log n \cdot f(d) - \sum_{d|n} \log d \cdot \mu(d)$$

$$\Lambda(n) = -\sum_{d|n} \log d\, \mu(d) \quad [\because n=1 \ \log n \cdot f(n) = 0$$

$$n > 1 \ \log n \cdot f(n) = 0]$$

Multiplicative function:- An arithmetic function 'f' is said to be multiplicative if $f(mn) = f(m) \cdot f(n)$ where $G.C.D(m,n) = 1$ and $G.C.D(m,n)$ are integers. 'f' is non-identically zero.

Completely Multiplicative function:- A multiplicative function 'f' is said to be completely multiplicative function $f(mn) = f(m) \cdot f(n)$

Result: If $f_\alpha(n)$ where 'α' is any real complex function then the function is completely multiplicative.

Proof: Given that $f_\alpha(n) = n^\alpha$

Consider $f_\alpha(mn) = (mn)^\alpha$

$= m^\alpha \cdot n^\alpha$

$= f_\alpha(m) \cdot f_\alpha(n)$

$\therefore f_\alpha$ is completely multiplicative

2. P.T Identity function is completely multiplicative

We know that $I(n) = \frac{1}{n}$

consider $I(mn) = \frac{1}{mn} = \left[\frac{1}{m}\right]\left[\frac{1}{n}\right]$

$= I(m) \cdot I(n)$

3. If 'f' is a multiplicative function then $f(1) = 1$

Given that 'f' is a multiplicative function

i.e., $f(n) = f(n \cdot 1) = f(n) \cdot f(1)$

$f(n) = f(n) \cdot f(1) = 0$

$f(n)[1 - f(1)] = 0$

$1 - f(1) = 0$

$f(1) = 1$

Since 'f' is a multiplicative function and

$f(n) \neq 0$

$\therefore 1 - f(1) = 0$

**Theorem II:- Give 'f' with $f(1) = 1$ then

(a) f is multiplicative iff

$f(P_1^{a_1} P_2^{a_2} P_3^{a_3} \ldots P_s^{a_s}) = f(P_1^{a_1}) \cdot f(P_2^{a_2}) \ldots f(P_s^{a_s})$

where P_i's are all primes and all integers $a_i \geq 1$

(b) f is multiplicative then f is completely multiplicative $\Leftrightarrow f(p^a) = [f(p)]^a$ where 'p' is a prime and $a \geq 1$

Proof:-

(a) Suppose 'f' is multiplicative

claim:- $f(P_1^{a_1} P_2^{a_2} \cdots P_s^{a_s}) = f(P_1^{a_1}) f(P_2^{a_2}) \cdots f(P_s^{a_s})$

consider $f(P_1^{a_1} P_2^{a_2} \cdots P_s^{a_s})$

$= f(P_1^{a_1}) [f(P_2^{a_2} \cdots P_s^{a_s})]$

$= f(P_1^{a_1}) f(P_2^{a_2}) [f(P_3^{a_3} \cdots P_s^{a_s})]$

$= f(P_1^{a_1}) f(P_2^{a_2}) f(P_3^{a_3}) \cdots f(P_s^{a_s})$

Conversely suppose that

$f(P_1^{a_1} P_2^{a_2} \cdots P_s^{a_s}) = f(P_1^{a_1}) f(P_2^{a_2}) \cdots f(P_s^{a_s})$

claim:- 'f' is multiplicative

i.e., $f(mn) = f(m) \cdot f(n)$ where $(m,n) = 1$

choose $m = P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r}$, $n = P_1^{a_1} P_2^{a_2} \cdots P_s^{a_s}$

consider $f(mn) = f(P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r} \cdot q_1^{a_1} q_2^{a_2} \cdots q_s^{a_r})$

$= f(P_1^{a_1}) f(P_2^{a_2}) \cdots f(P_r^{a_r}) f(q_1^{a_1}) f[q_2^{a_2} \cdots$

$= [f(P_1^{a_1}) f(P_2^{a_2}) \cdots f(P_r^{a_r})] [f(q_1^{a_1}) f(q_2^{a_2}) \cdots$

$= [f(P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r})] [f(q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r})]$

$= f(m) \cdot f(n)$

$\therefore$ 'f' is multiplicative

(b) Suppose 'f' is multiplicative

Assume 'f' is completely multiplicative

claim:- $f(p^a) = [f(p)]^a$

consider $f(p^a) = f(p\, p^{a-1})$

$= f(p) f(p^{a-1})$

$= f(p) f(p \cdot p^{a-2})$

$= f(p) \cdot f(p) \cdot f(p^{a-2})$

$= [f(p)]^2 f[p\, p^{a-3}]$

$$= [f(p)]^2 f(p) f(p^{a-3})$$

$$= [f(p)]^3 f(p^{a-3})$$

$$\vdots$$

$$= [f(p)]^a [f(p)]^0$$

$$= [f(p)]^a$$

$$\therefore f(p^a) = [f(p)]^a$$

Conversely suppose that 'f' is multiplicative and $f(p)^a = [f(p)]^a$

claim:- 'f' is completely multiplicative

i.e., p.T $f(mn) = f(m) \cdot f(n) \; \forall \; m, n$

choose $m = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$; $n = p_1^{b_1} p_2^{b_2} \cdots p_s^{b_s}$

consider $f(mn) = f(p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} \cdot p_1^{b_1} p_2^{b_2} \cdots p_s^{b_s})$

$$= f(p_1^{a_1+b_1} p_2^{a_2+b_2} \cdots p_s^{a_s+b_s})$$

$$= f(p_1^{a_1+b_1}) f(p_2^{a_2+b_2}) \cdots f(p_s^{a_s+b_s})$$

$$= [f(p_1^{a_1}) f(p_1^{b_1})][f(p_2^{a_2}) f(p_2^{b_2})] \cdots [f(p_s^{a_s}) f(p_s^{b_s})]$$

$$= [f(p_1^{a_1}) f(p_2^{a_2}) \cdots f(p_s^{a_s})][f(p_1^{b_1}) f(p_2^{b_2}) \cdots f(p_s^{b_s})]$$

$$= f[p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}] \cdot f[p_1^{b_1} p_2^{b_2} \cdots p_s^{b_s}]$$

$$= f(m) \cdot f(n)$$

$\therefore f(mn) = f(m) \cdot f(n) \; \forall \; m, n$

$\therefore$ 'f' is completely multiplicative.

sec:- Multiplicative function and Dirchilative function

Theorem:- If 'f' and 'g' are multiplicative then $f * g$ is also multiplicative.

Proof: Given that 'f' and 'g' are multiplicative.

We have 'f' is multiplicative.

i.e., $f(mn) = f(m)\cdot f(n)$ where $GCD(m,n) = 1$

We have 'g' is multiplicative

i.e., $g(mn) = g(m)g(n)$ where $GCD(m,n) = 1$

choose m,n such that $GCD(m,n) = 1$, $f * g(mn)$.

$$\sum_{d/mn} f(d)\, g\left[\frac{mn}{d}\right]$$

Every division of d of mn can be expressed as

$d = ab$ where a/m, b/n

$$f*g(mn) = \sum_{ab/mn} f(ab)\, g\left[\frac{mn}{ab}\right]$$

$$= \sum_{ab/mn} f(a)\cdot f(b)\, g\left[\frac{m}{a}\right] g\left[\frac{n}{b}\right]$$

$$= \sum_{a/m} f(a)\, g\left(\frac{m}{a}\right) \sum_{b/n} f(b)\, g\left(\frac{n}{b}\right)$$

$$= f*g(m)\cdot f*g(n)$$

$\therefore$ $f*g$ is multiplicative.

Theorem: If 'f' and 'g' are multiplicative then $f\cdot g$ and $\frac{f}{g}$ is also multiplicative.

Proof: Given that f and g are multiplicative

we have 'f' is multiplicative

i.e., $f(mn) = f(m)f(n)$ where $GCD(m,n) = 1$

we have 'g' is multiplicative

i.e., $g(mn) = g(m)g(n)$ where $GCD(m,n) = 1$

consider $f\cdot g(mn) = f[g(mn)]$

$= f[g(m)\cdot g(n)]$

$= f[g(m)]\cdot f[g(n)]$

$$= f \cdot g(m) * f \cdot g(n)$$

$$\frac{f}{g}(mn) = \frac{f(mn)}{g(mn)}$$

$$= \frac{f(m) f(n)}{g(m) g(n)}$$

$$= \frac{f(m)}{g(m)} \cdot \frac{f(n)}{g(n)}$$

$$= \frac{f}{g}(m) \cdot \frac{f}{g}(n)$$

$$\therefore \frac{f}{g}(mn) = \frac{f}{g}(m) \cdot \frac{f}{g}(n)$$

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Theorem:- If 'g' and $f*g$ are multiplicative then 'f' is multiplicative.

Proof:- Given that 'g' and '$f*g$' are multiplicative

claim:- 'f' is multiplicative

Now we can suppose that on the contrary way 'f' is not multiplicative

$\Rightarrow$ There exist a positive integers m, n with $GCD(m,n)=1$ such that $f(mn) \neq f(m) \cdot f(n)$

choose $h = f*g$ and $mn = 1$ then $h(mn) = h(1) = f*g(1)$

$$= \sum_{d/n} f(d) g\left[\frac{1}{d}\right]$$

$$h(1) = \sum_{d/n} f(1) g(1)$$

$$h(1) \neq 1$$

$\therefore h(1) \neq 1$

which is contradiction to 'h' is not a multiplicative

$\therefore$ 'f' is multiplicative

suppose $mn > 1$

let us suppose that $f(ab) = f(a) f(b)$ where $GCD(a,b)=1$ and $ab < mn$

On the contrary way, f is not multiplicative

consider $h(mn) = f*g(mn)$

$$= \sum_{d/mn} f(d)\, g\left[\frac{mn}{d}\right]$$

$$= \sum_{ab/mn} f(ab)\, g\left[\frac{mn}{ab}\right]$$

$$= \sum_{a/m} f(a)\, g\left(\frac{m}{a}\right) \sum_{b/n} f(b)\, g\left(\frac{n}{b}\right)$$

$$= f*g(m)\; f*g(n) + f(mn)$$

$h(mn) = h(m)h(n) + f(mn)$

$h(mn) = h(m)\,h(n) + f(mn) \neq 0$

$h(mn) - h(m)\,h(n) \neq f(mn)$

$h(mn) - h(m)h(n) \neq 0$

$\Rightarrow h(mn) \neq h(m)h(n)$

$\therefore$ h is not multiplicative

which is a contradiction

$\therefore$ 'f' is multiplicative

Theorem:- If 'g' is multiplicative then $\bar{g}^1$ is also multiplicative.

Proof:- Given that 'g' is multiplicative

we know that $g*\bar{g}^1 = I$ where 'I' is an identity function which is multiplicative

$\therefore g*\bar{g}^1$ is also multiplicative and g & $\bar{g}^1$ is multiplicative

$\therefore \bar{g}^1$ is multiplicative.

The inverse of completely multiplicative function

statement:- If 'f' is multiplicative then 'f' is completely multiplicative.

$\Leftrightarrow f^{-1}(n) = \mu(n) f(n) \quad \forall\ n \geq 1$

Proof:- Given that 'f' is multiplicative

Part ①:- Suppose 'f' is completely multiplicative

claim:- $f^{-1}(n) = \mu(n) f(n) \quad \forall\ n \geq 1$

let $g(n) = \mu(n) f(n) \quad \forall\ n \geq 1$

Now it is enough to P.T $f^{-1}(n) = g(n)$

i.e., we have to s.T $f * g(n) = I(n)$

consider $g * f(n) = \sum_{d|n} g(d) f\left(\frac{n}{d}\right)$

$$= \sum_{d|p} \mu(d) f(d) f\left[\frac{p}{d}\right]$$

$$= \sum_{d|p} \mu(d) f(d) \frac{f(p)}{f(d)}$$

$$= \sum_{d|p} \mu(d) f(p)$$

$$= f(p) \sum_{d|p} \mu(d)$$

$$= f(p) \cdot \frac{p}{1}$$

$$= f(p) \cdot I(p)$$

$$= f(1) \cdot I(p)$$

$$= 1 \cdot I(1)$$

$$= I(1)$$

$$g * f(n) = I(n)$$

$$g(n) = f^{-1}(n)$$

$$\therefore f^{-1}(n) = \mu(n) \cdot f(n)$$

Part ②:-

Conversely suppose that $f^{-1}(n) = \mu(n) f(n) \quad \forall\ n \geq 1$

claim:- 'f' is completely multiplicative

It is enough to s.T $f(pq) = [f(p)]^a$

we know that $f^{-1} * f(n) = I(n)$

$$\sum_{d|n} f^{-1}(d) \cdot f\left[\frac{n}{d}\right] = 0$$

$$\sum_{d|n} \mu(d) \cdot f(d) \frac{f(n)}{f(d)} = 0$$

$$\Rightarrow \sum_{d|n} \mu(d) \cdot f(d) \cdot f\left(\frac{n}{d}\right) = 0$$

Take $n = p^a$ and separate the divisior of p^a i.e., the divisioors is $1, p, p^2, \ldots p^a$

$$\mu(1) \cdot f(1) \cdot f\left[\frac{p^a}{1}\right] + \mu(p) \cdot f(p) \cdot f\left[\frac{p^a}{p}\right] = 0$$

$$f(p^a) - f(p) \cdot f(p^{a-1}) = 0$$

$$f(p^a) = f(p) \cdot f(p^{a-1})$$

$$= f(p) \cdot f[p\, p^{a-2}]$$

$$= f(p) \cdot f(p) \cdot f(p^{a-2})$$

$$= [f(p)]^2 f(p^{a-2})$$

$$= [f(p)]^2 \cdot f[p \cdot p^{a-3}]$$

$$\vdots$$

$$[f(p)]^a \cdot f[p^{a-1}]$$

$$[f(p)]^a$$

$$\therefore f(p^a) = [f(p)]^a$$

$\therefore$ 'f' is completely multiplicative [by above th]

Theorem:- If 'f' is multiplicative then $\sum_{d|n} \mu(d) \cdot f(d) = \prod_{p|n} [1 - f(p)]$

Proof:- Given that 'f' is multiplicative

Prove that $\sum_{d|n} \mu(d) \cdot f(d) = \prod_{p|n} [1 - f(p)]$

let $g(n) = \sum_{d|n} \mu(d) f(d)$

- First we have to p.T 'g' is multiplicative so determine $g(n)$ is sufficient to compute $g(p^a)$ for prime power

(i) consider LHS

$$g(mn) = \sum_{d|mn} f(d)\mu(d)$$

let $d = ab$

$$= \sum_{ab|mn} f(ab)\mu(ab)$$

$$= \sum_{ab|mn} f(a)f(b)\mu(a)\mu(b)$$

$$= \sum_{ab|mn} \mu(a)f(a) \sum_{b|n} \mu(b)f(b)$$

$$= g(m)g(n)$$

$$= R.H.S$$

$\therefore$ 'g' is multiplicative.

(ii) To prove that $g(p^a)$ for prime power

since $g(n) = \sum_{d|n} \mu(d) f(d)$

put $n = p^a$

$$g(p^a) = \sum_{d|p^a} \mu(d) f(d)$$

$$= \mu(p^a) + f(p^a) + \mu(p^1) - f(p^1)$$

$$\therefore g(p^a) = 1 - f(p)$$

for, $n > 1$ where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

$$g(n) = g\left[p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}\right]$$

$$= g(p_1^{a_1}) g(p_2^{a_2}) \cdots g(p_k^{a_k})$$

$= [1 - f(p_1)][1 - f(p_2)] \cdots [(1 - f(p_k))]$

$\therefore g(n) = \prod_{p/n} [1 - f(p)]$

$\therefore \sum_{d/n} \mu(d) f(d) = \prod_{p/n} [1 - f(p)]$

Hence proved.

Liouville's function $\lambda(n)$:-

The Liouville's function denoted by "$\lambda(n)$" defined $\lambda(1) = 1$ and if $n > 1$ where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ then

n	1	2	3	4	5	6	7	8	9	10
$\lambda(n)$	1	-1	-1	1	-1	1	-1	-1	1	1

Put $n = 2 \Rightarrow \lambda(2) = (-1)^1 = -1$

$n = 3 \Rightarrow \lambda(3) = (-1)^1 = -1$

$n = 4 \Rightarrow \lambda(4) = (-1)^2 = 1$

$n = 5 \Rightarrow \lambda(5) = (-1)^1 = 1$

$n = 6 \Rightarrow \lambda(6) = (-1)(-1) = 1$

$n = 7 \Rightarrow \lambda(7) = (-1)^1 = -1$

$n = 8 \Rightarrow \lambda(8) \Rightarrow \lambda(4)\lambda(2) = 1(-1) = -1$

$n = 9 \Rightarrow \lambda(9) = (-1)^2 = 1$

$n = 10 \Rightarrow \lambda(10) = \lambda(5)\lambda(2) = (-1)(-1) = 1$

Result:- λ is completely multiplicative

Proof:- To prove that 'λ' is completely multiplicative

i.e; $\lambda(mn) = \lambda(m)\lambda(n) \; \forall \; m, n$

There exist a +ve integers 'm and n'

let $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

$n = p^{b_1} p_2^{b_2} \cdots p_k^{b_k}$

consider L.H.S

i.e., $\lambda(mn) \cdot \lambda(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \cdot p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k})$

$= (-1)^{a_1+a_2+\cdots+a_k+b_1+b_2+\cdots+b_k}$

$= (-1)^{a_1+a_2+\cdots+a_k} (-1)^{b_1+b_2+\cdots+b_k}$

$= \lambda(m) \cdot \lambda(n)$

$= R.H.S$

$\lambda(mn) = \lambda(m)\lambda(n)$

$\therefore$ 'λ' is completely multiplicative

Hence proved.

Theorem:- For every $n \geq 1$ then $\sum_{d|n} \lambda(d) = \begin{cases} 1 \text{ if } n \text{ is square} \\ 0, \text{otherwise} \end{cases}$

Proof:- let $g(n) = \sum_{d|n} \lambda(d)$

First we have to Prove that 'g' is multiplicative

$g(n)$ we need to compute $g(p^a)$ for prime powers

To prove that 'g' is multiplicative

i.e., $g(mn) = g(m) \cdot g(n)$

L.H.S:- $g(mn) = \sum_{d|mn} \lambda(d)$ let $d = ab$

$g(mn) = \sum_{ab|mn} \lambda(ab)$

$= \sum_{a|m} \lambda(a) \sum_{b|n} \lambda(b)$

$= g(m) \cdot g(n)$

$\therefore$ 'g' is multiplicative.

To prove $g[p^a]$ for prime:-

since $g(n) = \sum_{d|n} \lambda(d)$

put $n = p^a$

$$g(p^a) = \sum_{d|p^a} \lambda(d)$$

$$= \lambda(p^0) + \lambda(p^1) + \lambda(p^2) + \cdots$$

$$= \lambda(1) + (-1) + (-1)^2 + \cdots$$

$$= 1 - 1 + 1 + \cdots$$

$$g(p^a) = \begin{cases} 1 & \text{if } a \text{ is even} \\ 0 & \text{if } a \text{ is odd} \end{cases}$$

Put $n = 1$:-

$$g(n) = \sum_{d|n} \lambda(d)$$

$$g(n) = \sum_{d|1} \lambda(1)$$

$$= \lambda(1) \quad [\because d = 1]$$

Put $n > 1$:-

where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

$$g(n) = g\left[p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}\right]$$

$$= g(p_1^{a_1})\, g(p_2^{a_2}) \cdots g(p_k^{a_k})$$

$$\Rightarrow g(n) = \begin{cases} 1 & \text{if } a_k \text{ is even} \\ 0 & \text{if } a_k \text{ is odd} \end{cases}$$

$$\therefore g(n) = \begin{cases} 1 & \text{if } n \text{ is square} \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

Division functions:-

The division function denoted by $\sigma_\alpha(n)$ for real (or) complex α and any integer $n \geq 1$, we $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$; The sum of the '$\alpha$' powers of the divisors of 'n'.

The function σ_α is called Division function.

Examples:-

1. $\sigma_2(4) = \sum_{d|4} d^2 = 1+2^2+4^2 = 21$

2. $\sigma_2(6) = \sum_{d|6} d^2 = 1^2+2^2+3^2+6^2 = 50$

3. $\sigma_{-1}(2) = \sum_{\substack{d|2 \\ (-1,2)}} d^{-1} = (1)^{2^{-1}} + 2^{2^{-1}} = 1 + \frac{1}{2} = \frac{3}{2}$

$= \sum_{\substack{d|2 \\ (1,2)}} d^{-1}$

Note:- If $\alpha = 0$ then $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$

$\sigma_0(n) = \sum_{d|n} d^0 = \sum_{d|n} (1)$ is the number of divisions of 'n'

It is denoted by $d(n)$.

Result:- To prove $\sigma_\alpha(n) = (\mu * N^\alpha)(n)$.

Proof:- We know that $\sigma_\alpha(n) = \sum_{d|n} 1 \cdot d^\alpha$

$= \sum_{d|n} 1 \cdot d^\alpha$

$= \sum_{d|n} \mu\left(\frac{n}{d}\right) d^\alpha$

$= \sum_{d|n} \mu\left(\frac{n}{d}\right) N^\alpha(d)$

$= (\mu * N^\alpha)(n)$

$\therefore \sigma_\alpha(n) = (\mu * N^\alpha)(n)$.

Theorem:- For $n \geq 1$, then $\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d) \mu\left(\frac{n}{d}\right)$

we know that $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$

$= \sum_{d|n} 1 \cdot d^\alpha$

$= \sum_{d|n} \mu\left(\frac{n}{d}\right) d^\alpha$

$= \sum_{d|n} \mu\left(\frac{n}{d}\right) N^\alpha(d)$

$$\sigma_\alpha(n) = (\mu * N^\alpha)(n)$$

$$\sigma_\alpha^{-1}(n) = (\mu * N^\alpha)^{-1}(n)$$

$$= (N^\alpha)^{-1}(n) * \mu^{-1}(n)$$

$$= [\mu(n) N^\alpha(n)] * \mu(n) \quad [\because \mu^{-1}(n) = \mu(n)]$$

$$= (\mu N^\alpha * \mu)(n)$$

$$= \sum_{d|n} (\mu N^\alpha)(d)\, \mu\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d) N^\alpha(d)\, \mu\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d) d^\alpha \mu\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} d^\alpha \mu(d) \mu\left(\frac{n}{d}\right)$$

Generalised Convolutions:- let 'f' be a real valued complex functions denoted on the positive real axis $(0, +\infty)$ such that $f(x) = 0$ for $0 \le x$

let 'α' is any arithmetic function then the convolution 'α' denoted on $[0, \infty]$ and is given by $\alpha \circ f(x) = \sum_{n \le x} \alpha(n) f\left(\frac{x}{n}\right)$

Note:- Convolution is generalised to dirchilet convolution

i.e; $\alpha \circ f(n) = (\alpha * f)(x)$ where α, f are arithmetic functions.

Theorem:- Associative property Relation 'o' and '*' for any arithmetic function 'α' and 'β' we have $\alpha \circ (\beta \circ f) = (\alpha * \beta) \circ F$

Proof:- We know that $\alpha \circ f(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$

claim:- $\alpha \circ (\beta \circ F) = (\alpha * \beta) \circ F$

consider $\alpha \circ (\beta \circ F)(x) = \sum \alpha(n)\, \beta \circ F\left(\frac{x}{n}\right)$

$$= \sum_{n \le x} \alpha(n) \sum_{m \le x/n} \beta\left(\frac{x}{n}\right) F\left(\frac{x/n}{m}\right) \left[\frac{x}{x/n}\right]$$

$$= \sum_{mn \le x} \alpha(n)\beta(m) F\left(\frac{x}{mn}\right)$$

$$= \sum_{k \le x} \sum_{n/k} \alpha(n)\beta\left(\frac{k}{n}\right) F\left(\frac{x}{k}\right)$$

$$= [(\alpha * \beta) \circ F](x)$$

$$\therefore \alpha \circ (\beta \circ F) = (\alpha * \beta) \circ F$$

Generalised Inversion formula:- If 'α' has a dirchilet inverse α^{-1} then the equation $g(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right) \Leftrightarrow F(x) = \sum_{n \le x} \alpha^{-1}(n) g\left(\frac{x}{n}\right)$

Proof:- $G(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$

$$G(x) = (\alpha \circ F)(x)$$

$$= \alpha \circ F$$

$$\alpha^{-1} \circ G = \alpha^{-1} \circ (\alpha \circ F)$$

$$\alpha^{-1} \circ G = (\alpha^{-1} \circ \alpha) \circ F$$

$$= I \circ F$$

$$= F$$

$$\therefore \alpha^{-1} \circ G(n) = F$$

$$\therefore F(x) = \sum_{n \le x} \alpha^{-1}(n) g\left(\frac{x}{n}\right)$$

Conversely, let $F(x) = \sum_{n \le x} \alpha^{-1}(n) g\left(\frac{x}{n}\right)$

$$\Rightarrow F(x) = \alpha^{-1} \circ G(x)$$

$$\alpha \circ F(x) = \alpha \circ (\alpha^{-1} \circ G)(x)$$

$$= (\alpha \circ \alpha^{-1}) \circ G(x)$$

$$= I \circ G(x)$$

$$= G(x)$$

$$\alpha \circ F = G$$

$$\therefore G(x) = \alpha \circ F(x)$$

$$= \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$$

$$G(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$$

Generalised Mobius inversion formula: If 'α' is completely multiplicative then $G(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$

$$\Leftrightarrow F(x) = \sum_{n \le x} \alpha(n)\mu(n)\, G\left(\frac{x}{n}\right)$$

Proof: Given that 'α' is completely multiplicative Then by known theorem we have $\alpha^{-1}(n) = \mu(n)\alpha(n)$

Assume that $G(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$

$$= \alpha \circ F(x)$$

$$= \alpha \circ F$$

$$\Rightarrow \alpha^{-1} \circ G = \alpha^{-1} \circ (\alpha \circ F)$$

$$= (\alpha^{-1} \circ \alpha) \circ F$$

$$= I \circ F$$

$$= F$$

$$\therefore \alpha^{-1} \circ G(\alpha) = F(x)$$

Now $F(x) = \sum_{n \le x} \alpha^{-1}(n) G\left(\frac{x}{n}\right)$

$$= \sum_{n \le x} \alpha(n)\mu(n) G\left(\frac{x}{n}\right)$$

$$= \sum_{n \le x} \alpha(n)\mu(n) G\left(\frac{x}{n}\right)$$

Conversely suppose that $F(x) = \sum_{n \le x} \alpha(n)\mu(n) G\left(\frac{x}{n}\right)$

$$= \sum_{n \le x} \alpha^{-1}(n) G\left(\frac{x}{n}\right)$$

$$= (\alpha^{-1} \circ G)(x)$$

$$\alpha^{-1} \circ G = F$$

$$\alpha \circ (\alpha^{-1} \circ G) = \alpha \circ F$$

$$(\alpha \circ \alpha^{-1}) \circ G = \alpha \circ F$$

$$I \circ G = \alpha \circ F$$

$$G = \alpha \circ F$$

$$\therefore G(x) = \alpha \circ F(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$$

$$\therefore G(x) = \sum_{n \le x} \alpha(n) F\left(\frac{x}{n}\right)$$

Hence proved.

Theorem:- If $n \ge 1$ then we have $\sum_{d|n} \phi(d) = n$

proof:- let 's' denote the set $\{1, 2, 3 \cdots\}$ distribate the integers 's' into disjoint sets

For each division 'd' of n, $A(d) = \{k \mid (k, n) = d, 1 \le k \le n\}$

clearly the set A(d) forms a disjoint union of 's'.

let d_1, d_2 are the disjoint divisions of 'n'

Suppose $A(d_1) = A(d_2)$

let $k \in A(d_1) = A(d_2)$

$\Rightarrow (k, n) = d_1$ and $(k, n) = d_2$

$\Rightarrow d_1 = d_2$, which is a contradiction

$\therefore A(d_1) = A(d_2)$

Let f(d) denote the number of integers in A(d)

then we have $\sum_{d|n} f(d) = n$ ——— (1)

But for $k \in A(d)$ and $(k, n) = d \Rightarrow \left(\frac{k}{d}, \frac{n}{d}\right) = 1$ and

also $1 \le k \le n$

$\Rightarrow 0 \le k \le n$

$\Rightarrow 0 < \frac{k}{d} \le \frac{n}{d}$

$\Rightarrow 0 < c \le n/d$

$\Rightarrow 1 \le e \le \frac{n}{d}$

$\therefore (e, \frac{n}{d}) = 1$ and $1 \le e \le \frac{n}{d}$

The number of such equations are given by

$$\sum_{\substack{e=1 \\ (e, \frac{n}{d})=1}}^{n/d} 1 = \phi\left(\frac{n}{d}\right)$$

Here $f(d) = \phi\left(\frac{n}{d}\right)$

$$\Rightarrow \sum_{d|n} f(d) = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

$$n = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

$$\therefore \sum_{d|n} \phi(d) = n$$

Hence proved.

Problems:-

1. Mobius function is multiplicative but not completely multiplicative.

First we P.T 'M' is multiplicative

i.e, $\mu(nm) = \mu(m)\mu(n)$ where $(m,n) = 1$

If either m (or) n has a square factors then mn is also a square factor

$\therefore \mu(n) = 0, \mu(m) = 0, \mu(nm) = 0$

$\therefore$ L.H.S = R.H.S

$\rightarrow$ If both 'm' and 'n' has no square factors and G.C.D of $(m,n) = 1$ then

$m = P_1 P_2 \cdots P_s \Rightarrow \mu(m) = (-1)^s$

$n = P_1 P_2 \cdots P_r \Rightarrow \mu(n) = (-1)^r$

LHS $\mu(mn) = \mu(P_1 P_2 \cdots P_s . P_1 P_2 \cdots P_r)$

$= (-1)^{r+s}$

$= (-1)^r (-1)^s = \mu(m) \cdot \mu(n)$

$\therefore \mu(mn) = \mu(m)\cdot\mu(n)$ where $(m,n) = 1$

Hence Mobius function is multiplicative, but not completely multiplicative.

Example:- $\mu(18) = (-1)^1 \mu(2) = (-1)^1$

$\mu(18\cdot 2) = \mu(36) = \mu(6)^2 = 0$

$\therefore \mu(18\cdot 2) \neq \mu(18)\cdot\mu(2)$

$\therefore \mu(mn) \neq \mu(m)\cdot\mu(n)$

2. Euler Quotient function is multiplicative but not completely multiplicative.

First we P.T ϕ is multiplicative

i.e, $\phi(mn) = \phi(m)\cdot\phi(n)$ whenever $(m,n) = 1$

we know that by known Theorem, $\phi(mn) = \phi(m)\cdot\phi(n)\cdot\frac{d}{\phi(d)}$

whenever $(m,n) = 1$

Take $(m,n) = 1 \Rightarrow d = 1$

L.H.S $\phi(mn) = \phi(m)\cdot\phi(n)\cdot\frac{1}{\phi(1)}$

$= \phi(m)\cdot\phi(n)$

$\therefore$ Euler totient function is multiplicative but not completely multiplicative.

Example:- $\phi(8) = \phi(2^3) = 2^3\left[1-\frac{1}{2}\right]$

$= 4$

$\phi(2)\cdot\phi(4) = 2^{-1}\left[2^2\left(1-\frac{1}{2}\right)\right] = 2$

$\therefore \phi(8) \neq \phi(2)\cdot\phi(4)$